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Non-Crossing Parametric Quantile Functions: An Application to Extreme Temperatures

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Contents

Contents

Background

Let Y be a response variable, and $\boldsymbol{x}$ a p-dimensional vector of covariates. The conditional quantile function could be written as:

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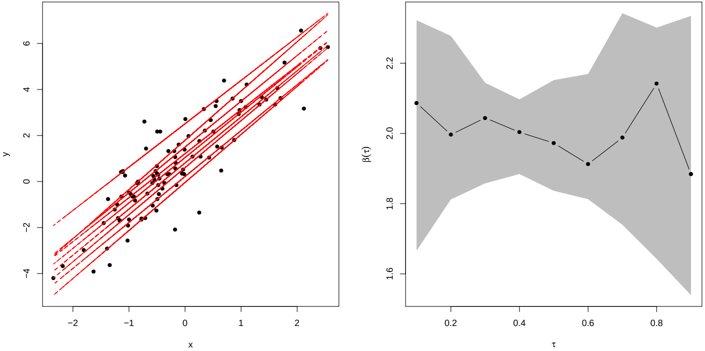
- $Q(\tau \mid \boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{\beta}(\tau)$ in **quantile regression** (QR, Koenker and Bassett Jr, 1978);

Background

Let Y

cova



Problems:

- Quantiles are estimated one at a time
- The estimated regression coefficients are non-smooth functions of τ

Background

Let Y be a response variable, and $\boldsymbol{x}$ a p-dimensional vector of covariates. The conditional quantile function could be written as:

- $Q(\tau \mid \boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{\beta}(\tau)$ in **quantile regression** (QR, Koenker and Bassett Jr, 1978);
- $Q(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T\boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}) = \boldsymbol{x}^T\boldsymbol{\theta}\boldsymbol{b}(\tau)$ in **quantile regression coefficients modelling** (QRCM, Frumento and Bottai, 2016).

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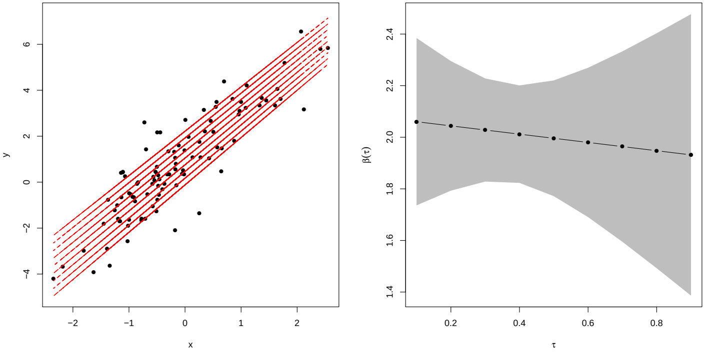

Pros:

- Permits modelling the entire quantile function
- The estimated regression coefficients are smooth functions of τ
- Quantile crossing can be *quantified* and *eliminated*

Background

Let Y be a response variable, and $\boldsymbol{x}$ a p-dimensional vector of covariates. The conditional quantile function could be written as:

- $Q(\tau \mid \boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{\beta}(\tau)$ in **quantile regression** (QR, Koenker and Bassett Jr, 1978);
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General definition of crossing

Quantile crossing occurs when the estimated quantiles do not form a monotonically increasing function. For example, the fitted median is larger than the 51st percentile.

Let Y be a response variable, and $\boldsymbol{x}$ a p-dimensional vector of

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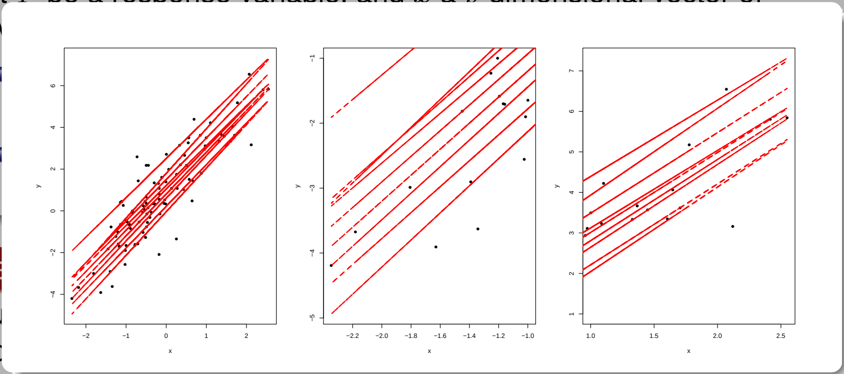

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larger than the 51st percentile.

Contents

A parametric quantile function

Linear effect of covariates

A parametric quantile function

Linear effect of covariates

You can define a parametric model for the coefficient functions, $\boldsymbol{\beta}(\tau) = \{\beta_1(\tau), \ldots, \beta_p(\tau)\}$. A convenient parametrization:

$$\beta_j(\tau \mid \boldsymbol{\theta}) = \theta_{j1} b_1(\tau) + \cdots + \theta_{jk} b_k(\tau)$$

$$\Downarrow$$

$$\boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}) = \boldsymbol{\theta} \boldsymbol{b}(\tau)$$

where $\boldsymbol{\theta}$ is a $p \times k$ matrix of unknown parameters.

A parametric quantile function

Linear effect of covariates

$$Q(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\theta} \boldsymbol{b}(\tau)$$

A parametric quantile function

Linear effect of covariates

$$Q(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\theta} \boldsymbol{b}(\tau)$$

$$\Downarrow$$

Example 1: Linear functions of τ

$$\beta_0(\tau) = \theta_{00} + \theta_{01}\tau$$

$$\beta_1(\tau) = \theta_{10} + \theta_{11}\tau$$

$$\boldsymbol{b}(\tau) = \begin{pmatrix} 1 \\ \tau \end{pmatrix} \text{ and } \boldsymbol{\theta} = \begin{pmatrix} \theta_{00} & \theta_{01} \\ \theta_{10} & \theta_{11} \end{pmatrix}$$

A parametric quantile function

Linear effect of covariates

$$Q(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\theta} \boldsymbol{b}(\tau)$$

Ex

A new interpretation:

- $\theta_0 + \theta_1 \tau$ is the quantile function of a $U(\theta_0, \theta_0 + \theta_1)$
- $Q(\tau \mid x, \boldsymbol{\theta})$ well defined if $\theta_{01} + \theta_{11} x > 0$ for all x
- If $\theta_{00} = \theta_{01} = 0$ (no intercept), a zero-inflated model

$$\boldsymbol{b}(\tau) = \begin{pmatrix} 1 \\ \tau \end{pmatrix} \text{ and } \boldsymbol{\theta} = \begin{pmatrix} \theta_{00} & \theta_{01} \\ \theta_{10} & \theta_{11} \end{pmatrix}$$

A parametric quantile function

Linear effect of covariates

$$Q(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\theta} \boldsymbol{b}(\tau)$$

$$\Downarrow$$

Example 2: A “mix” between Uniform and Normal

$$\beta_0(\tau) = \theta_{00} + \theta_{01} z(\tau)$$

$$\beta_1(\tau) = \theta_{10} + \theta_{12} \tau$$

$$\boldsymbol{b}(\tau) = \begin{pmatrix} 1 \\ z(\tau) \\ \tau \end{pmatrix} \text{ and } \boldsymbol{\theta} = \begin{pmatrix} \theta_{00} & \theta_{01} & 0 \\ \theta_{10} & 0 & \theta_{12} \end{pmatrix}$$

A parametric quantile function

Linear effect of covariates

$$Q(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\theta} \boldsymbol{b}(\tau)$$

$$\Downarrow$$

Ex

A new interpretation:

- $z(\tau)$ is the quantile function of a standard Normal
- we can estimate a standard linear model if we consider $\theta_{12} = 0$, $\beta_0 = \theta_{00}$, $\beta_1 = \theta_{10}$, and $\sigma = \theta_{01}$

$$\boldsymbol{b}(\tau) = \begin{pmatrix} 1 \\ z(\tau) \\ \tau \end{pmatrix} \text{ and } \boldsymbol{\theta} = \begin{pmatrix} \theta_{00} & \theta_{01} & 0 \\ \theta_{10} & 0 & \theta_{12} \end{pmatrix}$$

A parametric quantile function

Linear effect of covariates

$$Q(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\theta} \boldsymbol{b}(\tau)$$

$$\Downarrow$$

Example 3: Alternative model specifications

Polynomials: $\boldsymbol{b}(\tau) = [\tau, \tau^2, \ldots, \tau^k]^T$

Quantile function of standard Normal: $\boldsymbol{b}(\tau) = z(\tau)$

Quantile function of shifted Logistic: $\boldsymbol{b}(\tau) = [\log(\tau), \log(1-\tau)]^T$

Combination of trigonometric functions: $\boldsymbol{b}(\tau) = [\cos(\tau), \sin(\tau)]^T$

Piecewise linear

Quantile regression coefficients modelling

The estimator

Ordinary quantile regression for the τth quantile minimise

$$L(\boldsymbol{\beta}(\tau)) = \sum_{i=1}^{n} \rho_\tau(y_i - \boldsymbol{x}_i^T \boldsymbol{\beta}(\tau)) \quad (1)$$

where $\rho_\tau(u) = (\tau - I(u \leq 0))u$.

Quantile regression coefficients modelling

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where $\rho_\tau(u) = (\tau - I(u \leq 0))u$.

Frumento and Bottai, 2016 propose to estimate the unknown parameters $\boldsymbol{\theta}$ as the minimiser of

$$L(\boldsymbol{\theta}) = \sum_{i=1}^{n} \int_0^1 \rho_\tau(y_i - \boldsymbol{x}_i^T \boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}))\mathrm{d}\tau \qquad (2)$$

Quantile regression coefficients modelling

The estimator

Ordinary quantile regression for the τth quantile minimise

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Properties:

- Average loss function
- Estimating "all" quantiles at once
- Smooth loss function (simple computation and asymptotics)
- You can take the integral over (p_1, p_2) instead of $(0, 1)$
- More parsimonious and efficient than QR

$$L(\boldsymbol{\theta}) = \sum_{i=1} \int_0 \rho_\tau(y_i - \boldsymbol{x}_i^T \boldsymbol{\beta}(\tau \mid \boldsymbol{\theta}))\mathrm{d}\tau \quad (2)$$

Contents

Crossing in parametric quantile functions

The parametric structure of QRCM induces the non-crossing property.

Crossing in parametric quantile functions

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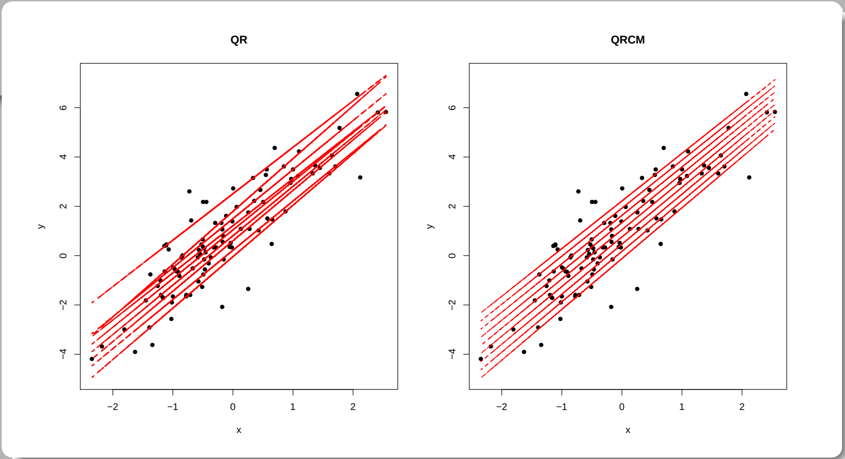

Crossing in parametric quantile functions

The parametric structure of QRCM induces the non-crossing property.

Definition

Assume to estimate a quantile function $Q(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\theta} \boldsymbol{b}(\tau)$, and denote by $Q'(\tau \mid \boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^T \boldsymbol{\theta} \boldsymbol{b}'(\tau)$ its first derivative. Denote by $\hat{\boldsymbol{\theta}}$ the minimiser of the loss function defined in equation (2). Quantile crossing occurs if the set $\{\tau : Q'(\tau \mid \boldsymbol{x}, \hat{\boldsymbol{\theta}}) < 0\}$ is non-empty.

Crossing in parametric quantile functions

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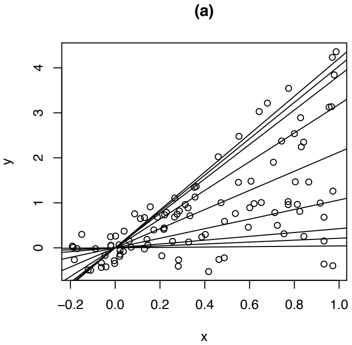

Figure 1: **Example (a). Misspecified model with empty feasible region.** Consider $Q(\tau \mid x, \theta) = \theta\tau x$. All regression lines cross at $x = 0$, where the model is assumed to be degenerated. If x only takes either positive or negative values, this model is guaranteed to be non-crossing. Otherwise, crossing occurs at all values of θ and cannot be avoided.

Crossing in parametric quantile functions

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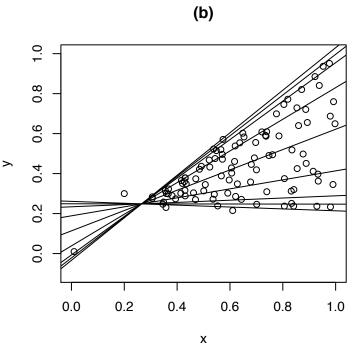

Figure 2: **Example (b). Crossing caused by outliers**. Define $Q(\tau \mid x, \boldsymbol{\theta}) = \beta_0(\tau \mid \boldsymbol{\theta}) + \beta_1(\tau \mid \boldsymbol{\theta})x$ with $\beta_0(\tau \mid \boldsymbol{\theta}) = \theta_{00} + \theta_{01}\tau$ and $\beta_1(\tau \mid \boldsymbol{\theta}) = \theta_{10} + \theta_{11}\tau$, and assume that x lies in the $(0, 1)$ interval. A simple condition for monotonicity is that $(\hat{\theta}_{01}, \hat{\theta}_{11}) \geq 0$. However, as shown, the estimated regression lines may cross in correspondence of outlying values.

Crossing in parametric quantile functions

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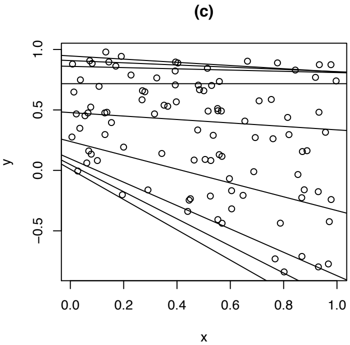

Figure 3: **Example (c). Non-monotone coefficient**. Define $Q(\tau \mid x, \boldsymbol{\theta}) = \theta_0\tau - \theta_1(\tau - 0.75)^2 x$, and assume $\boldsymbol{\theta} > \mathbf{0}$ and $x \geq 0$. The coefficient $\beta_1(\tau \mid \boldsymbol{\theta}) = -\theta_1(\tau - 0.75)^2$ associated with x is assumed to be a non-monotone function, and may obviously induce crossing. A non-crossing quantile function can be obtained by imposing $2\hat{\theta}_0 \geq \hat{\theta}_1 \max_i(x_i)$.

Crossing in parametric quantile functions

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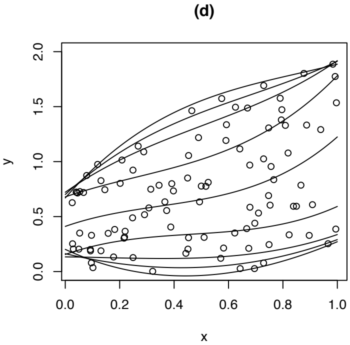

Figure 4: **Example (d). Crossing in a flexible model**. In most situations, the true model is not known and the coefficients can be described by smooth flexible functions. Consider, to model $\beta_j(\tau \mid \boldsymbol{\theta}) = \theta_{j0} + \theta_{j1}\tau + \theta_{j2}\tau^2 + \theta_{j3}\tau^3$, $j = 0, \ldots, 3$ and $Q(\tau \mid x, \boldsymbol{\theta}) = \beta_0(\tau \mid \boldsymbol{\theta}) + \beta_1(\tau \mid \boldsymbol{\theta})x + \beta_2(\tau \mid \boldsymbol{\theta})x^2 + \beta_3(\tau \mid \boldsymbol{\theta})x^3$. Crossing at extreme quantiles arises from the combination between a very flexible parametric structure and a relatively small sample size.

Contents

The constrained estimator

A parametric quantile function with the non-crossing property

$$\min L(\boldsymbol{\theta}) \tag{3}$$

$$\text{s.t.} \quad \int_0^1 |\min\{0, Q'(\tau \mid \boldsymbol{x}_i, \boldsymbol{\theta})\}| \mathrm{d}\tau = 0, \quad i = 1, \ldots, n.$$

The constrained estimator

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s.t. $\int_0^1 |\min\{0, Q'(\tau \mid \boldsymbol{x}_i, \boldsymbol{\theta})\}| \mathrm{d}\tau = 0, \quad i = 1, \ldots, n.$

$$\Downarrow$$

A penalised approach

$$\begin{aligned} L_\lambda(\boldsymbol{\theta}) &= L(\boldsymbol{\theta}) + \lambda P(\boldsymbol{\theta}) = \sum_{i=1}^{n} \int_0^1 \rho_\tau(y_i - Q(\tau \mid \boldsymbol{x}_i, \boldsymbol{\theta}))\mathrm{d}\tau \\ &+ \lambda \sum_{i=1}^{n} \int_0^1 |\min\{0, Q'(\tau \mid \boldsymbol{x}_i, \boldsymbol{\theta})\}| \mathrm{d}\tau. \end{aligned} \tag{4}$$

The constrained estimator

A parametric quantile function with the non-crossing property

$$\min L(\boldsymbol{\theta}) \qquad (3)$$

where:

- $\lambda > 0$, balances between the two ingredients of $L_\lambda(\boldsymbol{\theta})$, namely the unpenalised loss function, $L(\boldsymbol{\theta})$, and the penalty term, $P(\boldsymbol{\theta})$
- $P(\boldsymbol{\theta})$ is a penalty term computed as the sum of all constraints, and reflects both the sign and the absolute size of $Q'(\tau \mid \boldsymbol{x}_i, \boldsymbol{\theta})$

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$$+ \quad \lambda \sum_{i=1}^{n} \int_0^1 |\min\{0, Q'(\tau \mid \boldsymbol{x}_i, \boldsymbol{\theta})\}| \mathrm{d}\tau. \qquad (4)$$

Contents

Real data application

The data

We considered meteorological data from ECAD (European Climate Assessment and Dataset). The data are described in Klein et al. (2002) and can be downloaded from (https://www.ecad.eu/).

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Objective

We considered the minimum and the maximum daily temperature, particularly focusing on the extreme quantiles, corresponding to cold/heat waves.

Real data application

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Objective

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The model

$$Q(\tau \mid t, s) = \beta_0(\tau) + \boldsymbol{g}_t(t)^T \boldsymbol{\beta}_t(\tau) + \boldsymbol{g}_s(s)^T \boldsymbol{\beta}_s(\tau) + (\boldsymbol{g}_t(t) \otimes \boldsymbol{g}_s(s))^T \boldsymbol{\beta}_{t:s}(\tau) \quad (5)$$

Real data application

where:

- t is a progressive count $(1, 2, \ldots)$, expressed in days
- $s = (t \mod 365.2422)$ counts the days within a solar year (365.2422 days)
- $\boldsymbol{g}_t(t)$ model the long-term trend by using the basis of a restricted natural cubic spline, with one internal knot every 20 years
- $\boldsymbol{g}_s(s)$ model the seasonal variations by using a periodic spline, with a period of one solar year and one internal knot every 2 solar months
- $\boldsymbol{g}_t(t) \otimes \boldsymbol{g}_s(s)$ (the tensor product) defines an interaction term

Station n. 78

Kandalaksha, Russia

- Located at 26 meters above sea level
- Coordinates N 67° 09' 00", E 32° 21' 00"
- Time series from 1912 to 2018
- Temperature forecast up to 2050

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Table 1: Indicators of crossing using different methods for quantile regression. (QR, QRCM, QRCM$_c$). P_{cross} is computed as the proportion of observations for which the estimated quantile function was non-monotone; and L_{cross} is the average length of the crossing region on the τ scale.

		QR	QRCM	QRCM$_c$
Min temperatures	$100 \times P_{\text{cross}}$	45.03	2.94	0.00
	$100 \times L_{\text{cross}}$	2.69	0.23	0.00
Max temperatures	$100 \times P_{\text{cross}}$	34.56	0.32	0.00
	$100 \times L_{\text{cross}}$	2.56	0.03	0.00

Station n. 78

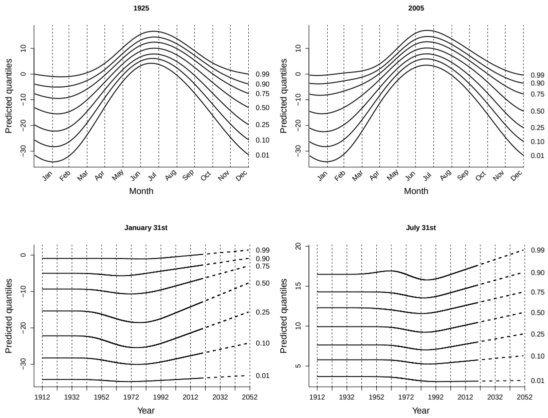

Figure 5: Estimated quantiles of order τ of the minimum daily temperature. The top panels compare the seasonal trend and bottom panels illustrate the long-term trends. Dashed lines indicate extrapolation.

Station n. 78

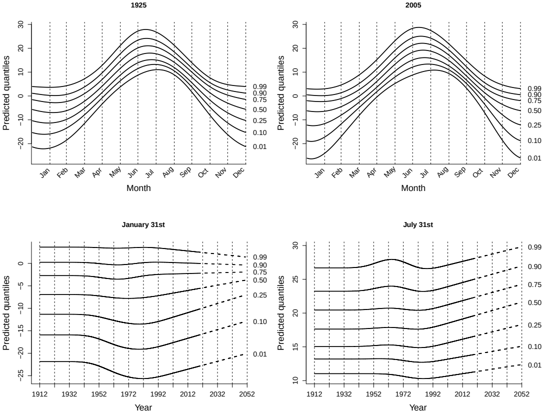

Figure 6: Estimated quantiles of order τ of the maximum daily temperature. The top panels compare the seasonal trend and bottom panels illustrate the long-term trends. Dashed lines indicate extrapolation.

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- Results on the climate change in Kandalaksha station highlighted a cooling between the 1960s and the early 1990s followed by a warming effect in the long-term trends and that we may expect in twenty years an additional warming effect well above 1°C in both winter and summer;

To sum-up

- We proposed a new parametric quantile function with the non-crossing property;
- We applied our proposal to estimate extreme quantiles through extrapolation;
- Results on the climate change in Kandalaksha station highlighted a cooling between the 1960s and the early 1990s followed by a warming effect in the long-term trends and that we may expect in twenty years an additional warming effect well above $1°$C in both winter and summer;
- A computationally efficient algorithm has been implemented in the `qrcm` package in R.

Thanks for the attention!!!